Quantum Enhancements of Involutory Birack Counting Invariants

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Abstract

The involutory birack counting invariant is an integer-valued invariant of unoriented tangles defined by counting homomorphisms from the fundamental involutory birack of the tangle to a finite involutory birack over a set of framings modulo the birack rank of the labeling birack. In this first of an anticipated series of several papers, we enhance the involutory birack counting invariant with *quantum weights*, which may be understood as tangle functors of involutory birack-labeled unoriented tangles.

KEYWORDS: Biracks, enhancements of counting invariants, quantum invariants

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1 Introduction

A *birack* $\mathbf{X}$ is an algebraic structure with axioms determined by the framed Reidemeister moves such that labelings of arcs in a tangle diagram with elements of $\mathbf{X}$ before and after a move are in bijective correspondence. In particular, the number of $\mathbf{X}$-labelings of a tangle diagram by a given involutory birack $\mathbf{X}$ is an invariant of unoriented framed links under framed isotopy moves. Biracks were introduced in [7]; special cases of biracks including racks, quandles and biquandles have been studied and used to define link invariants in various works including [6, 9, 12, 3, 11, 5] and more.

A finite birack has an associated integer N called the *birack rank* or *birack characteristic* with the property that framed unoriented links with equivalent framings modulo N have equal numbers of labelings. Summing over a complete period of framings, one obtains an integer-valued invariant of unframed tangles called the *integral birack counting invariant*, denoted $\Phi_{\mathbf{X}}^{\mathbb{Z}}$; see [13] for more.

An *enhancement* of $\Phi_{\mathbf{X}}^{\mathbb{Z}}$ is obtained by evaluating an invariant σ of $\mathbf{X}$-labeled diagrams on each $\mathbf{X}$-labeling of a tangle diagram T and collecting these values over a complete set of $\mathbf{X}$-labelings to obtain a multiset of σ-values sometimes called "signatures". This multiset is then a generally stronger invariant of unframed links whose cardinality is the birack counting invariant $\Phi_{\mathbf{X}}^{\mathbb{Z}}$.

In this paper we develop an enhancement of $\Phi_{\mathbf{X}}^{\mathbb{Z}}$ in the special case when $\mathbf{X}$ is an *involutory birack*, the type of birack appropriate for defining invariants of unoriented tangles, using σ values we call *quantum weights*. Quantum weights are birack-labeled tangle functors which may be understood as customized quantum invariants for $\mathbf{X}$-labeled tangle diagrams. This will be the first is a series of papers in which we consider progressively more complex cases, starting with the unoriented case in this paper and in the sequels considering the oriented and virtual cases. The paper is organized as follows. In Section 2 we review the basics of involutory biracks and the birack counting invariant. In Section 3 we introduce quantum weights and the quantum-enhanced birack counting invariant; we give examples to demonstrate how the invariant is computed and show that the enhanced invariant is stronger than the unenhanced invariant. In Section 4 we consider quantum enahancements of involutory birack counting invariants for closed braids. We conclude in Section 5 with some open questions and directions for future work.

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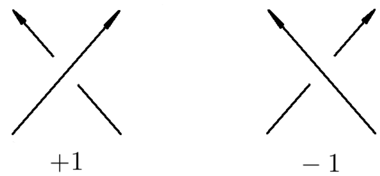
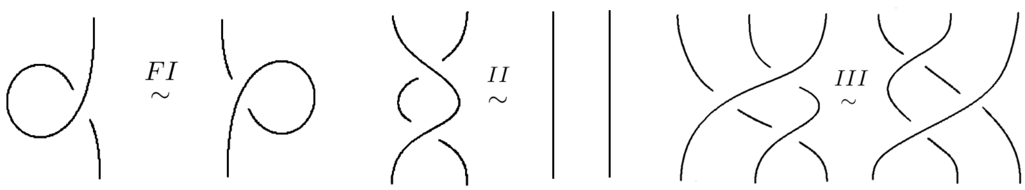
2 Involutory Biracks and the Counting Invariant

Recall that a *framed unoriented tangle* is an equivalence class of disjoint unions of simple closed curves and arcs in $A = \mathbb{R}^2 \times [0,1]$ with endpoints in fixed positions in ∂A; two such disjoint unions are equivalent if they can be connected by an ambient isotopy of A fixing ∂A and preserving the linking number of each component with a choice of framing curve for each component. The linking number of a component with its framing curve is the *framing number* of the component; the framing number is equal to the *writhe* or sum of crossing signs



at self-crossings when the framing curve is the *blackboard framing curve* obtained by pushing off a parallel copy of each component in a diagram of T.

Equivalently, framed unoriented tangles can be defined combinatorially as an equivalence class of unoriented tangle diagrams under the equivalence relation generated by the *framed unoriented Reidemeister moves*



In particular, the usual unframed Reidemeister I move changes the backboard framing of a component by ± 1, while the framed version depicted above preserves the framing. An unframed tangle of c components thus determines a $\mathbb{Z}^c$–lattice of framed tangles; a choice of ordering on the components allows us to specify a framing with a *framing vector* $\vec{w} \in \mathbb{Z}^c$.

Next, we have a definition from [2]:

Definition 1 An *involutory birack* is a set $\mathbf{X}$ with an invertible map $B : \mathbf{X} \times \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ such that

(i) $(\tau B)^2 = I$

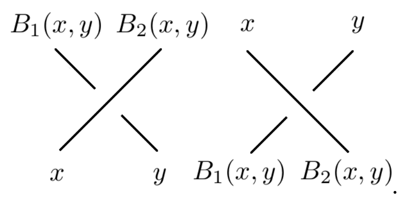
(ii) The components $(\tau B \Delta)_{1,2} : \mathbf{X} \to \mathbf{X}$ of the map $\tau B \Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ are bijections, and

(iii) B satisfies the *set-theoretic Yang-Baxter equation*

$$(B \times I)(I \times B)(B \times I) = (I \times B)(B \times I)(I \times B)$$

where $\tau : \mathbf{X} \times \mathbf{X} \to \mathbf{X} \times \mathbf{X}$, $\Delta : \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ and $I : \mathbf{X} \to \mathbf{X}$ are defined by $\tau(x, y) = (y, x)$, $\Delta(x) = (x, x)$ and $I(x) = x$ respectively. The map $S : \mathbf{X} \times \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ defined by $S = \tau B \tau = B^{-1}$ is called the *sideways map*. We will find it useful to abbreviate $B_1(x, y) = y^x$, $B_2(x, y) = x_y$, $(S\Delta)_2^{-1} = \alpha$ and $(S\Delta)_1 \alpha = \pi$.

Let $\mathbf{X}$ be an involutory birack. A *birack labeling* or $\mathbf{X}$–*labeling* of an unoriented framed tangle diagram T is an assignment of elements of $\mathbf{X}$ to the *semiarcs* of T, i.e., the portions of T between crossing points, such that at every crossing we have the pictured relationship between semiarc labelings:



The involutory birack axioms are then consequences of the framed Reidemeister moves; more precisely, they are the conditions required to guarantee that for every birack labeling of a tangle diagram before a framed unoriented Reidemeister move, there is exactly one corresponding labeling of the diagram T' after the Reidemeister move. In particular, given a finite involutory birack $\mathbf{X}$, the number of $\mathbf{X}$–labelings of a framed tangle T is an invariant of unoriented framed isotopy called the *basic birack counting invariant*, denoted $\Phi^B_{\mathbf{X}}(L)$.

Standard examples of involutory biracks include

- *Constant Action Involutory Biracks.* Let $\mathbf{X}$ be any set and $\sigma, \tau : \mathbf{X} \to \mathbf{X}$ two involutions such that $\sigma\tau = \tau\sigma$. Then
$$B(x,y) = (\sigma(y), \tau(x))$$
defines an involutory birack structure on $\mathbf{X}$.

- *Involutory (t,s,r)-Biracks.* Let $\mathbf{X}$ be a module over the ring $\tilde{\Lambda} = \mathbb{Z}[t,s,r]/I$ where I is the ideal generated by $s^2 - s(1-tr), 1-t^2, 1-r^2, (t+r)s$, and $(1-r)s$. Then $\mathbf{X}$ is an involutory birack with map
$$B(x,y) = (ty + sx, rx).$$

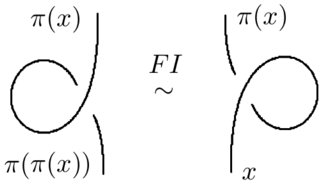
- *The Fundamental Involutory Birack of an Unoriented Framed Tangle.* Let T be an unoriented framed tangle and let $G = \{g_1, \dots, g_n\}$ be a set of generators corresponding bijectively with the semiarcs in T. The set W of *involutory birack words* in T is defined recursively by the rules (1) $x \in G \Rightarrow x \in W$ and (2) $x, y \in W \Rightarrow B_1(x,y) \in W$ and $B_2(x,y) \in W$. Then the *Fundamental Involutory Birack* of T, $IB(T)$, is the set of equivalence classes in W under the equivalence relation determined by the involutory birack axioms and the crossing relations in T. Note that birack labelings of a tangle T by a birack $\mathbf{X}$ are precsiely birack homomorphisms $f : IB(T) \to \mathbf{X}$, i.e. maps satisfying $(f \times f)B_{IB(T)} = B_{\mathbf{X}}(f \times f)$ where $B_{\mathbf{X}}$ and $B_{IB(T)}$ are the birack maps in $\mathbf{X}$ and $IB(T)$ respectively.

Given a finite set $\mathbf{X} = \{x_1, \dots, x_n\}$ we can define an involutory birack structure on $\mathbf{X}$ by giving a matrix $M_{\mathbf{X}} = [U|L]$ encoding the operation tables of the components of B considered as binary operations $B(x,y) = (y^x, x_y)$. That is, $M_{\mathbf{X}}$ is a block matrix with two blocks U and L such that the entries in row i, column j of U^T and L respectively are k and l where $B(x_i, x_j) = (x_k, x_l)$.[1] Such a matrix defines an involutory birack if and only if the map $B : \mathbf{X} \times \mathbf{X} \to \mathbf{X} \times \mathbf{X}$ defined by the matrix satisfies the birack axioms.

Example 1 The smallest involutory birack which is neither a rack nor a quandle is given by the birack matrix
$$\left[\begin{array}{cc|cc} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \end{array}\right].$$

[1] We use the transpose of U so that in an $\mathbf{X}$-labeling of a tangle diagram, the row label operand and the output label lie on the same strand.

For a birack $\mathbf{X}$, the *kink map* $\pi : \mathbf{X} \to \mathbf{X}$ defined by $\pi = (S\Delta)_1 (S\Delta)_2^{-1}$ is a bijection representing going through a positive kink as pictured.

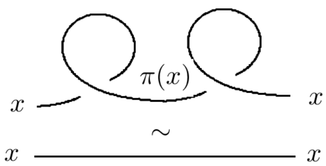


The exponent of π considered as an element of the symmetric group $S_{\mathbf{X}}$, i.e. the smallest positive integer N such that $\pi^N = I$, is called the *birack rank* or *birack characteristic* of $\mathbf{X}$. In the unoriented case, the framed type I move then requires that $\pi = \pi^{-1}$ and we obtain

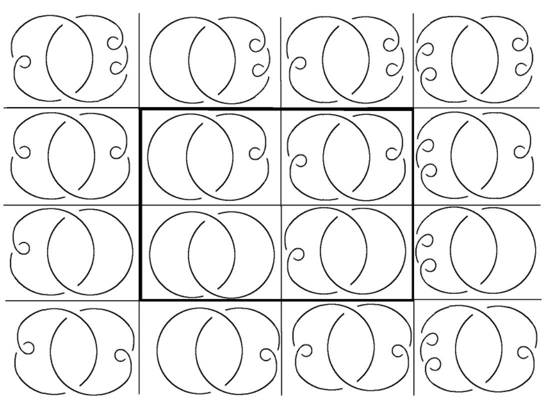
Theorem 1 *An involutory birack has birack rank $N = 1$ or $N = 2$.*

An involutory birack of rank $N = 1$ is an *involutory biquandle* or *bikei* (双圭).

By construction, if $\mathbf{X}$ is an involutory birack of rank N, then $\mathbf{X}$-labelings of a framed tangle diagram before and after the *N-phone cord move* are in bijective correspondence. When $N = 1$, the N-phone cord move is the unframed Reidemeister type I move; when $N = 2$, the move is as pictured below.



Given an unoriented framed tangle T of c components and a finite involutory birack $\mathbf{X}$, the $\mathbb{Z}^c$–lattice of unoriented framings of T determines a $\mathbb{Z}^c$–lattice of basic counting invariant values $\Phi^B_{\mathbf{X}}(T, \vec{w})$.



$\Phi^B_X(L,(-1,2))$	$\Phi^B_X(L,(0,2))$	$\Phi^B_X(L,(1,2))$	$\Phi^B_X(L,(2,2))$
$\Phi^B_X(L,(-1,1))$	$\Phi^B_X(L,(0,1))$	$\Phi^B_X(L,(1,1))$	$\Phi^B_X(L,(2,1))$
$\Phi^B_X(L,(-1,0))$	$\Phi^B_X(L,(0,0))$	$\Phi^B_X(L,(1,0))$	$\Phi^B_X(L,(2,0))$
$\Phi^B_X(L,(-1,-1))$	$\Phi^B_X(L,(0,-1))$	$\Phi^B_X(L,(1,-1))$	$\Phi^B_X(L,(2,-1))$

If $N = 1$, then these basic counting invariants are all equal; if $N = 2$, then any two framings of T with framing vectors congruent mod 2 have the same basic counting invariants, so the $\mathbb{Z}^c$–lattice is tiled with a 2×2 tile of basic counting invariant values. In either case, we can sum the basic counting invariants over an $N \times N$ tile of framings to obtain an invariant of unframed unoriented links T called the *integral involutory birack counting invariant*,

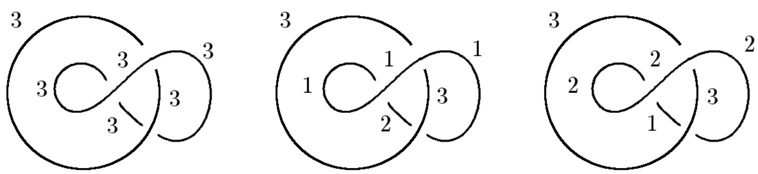
$$\Phi^{\mathbb{Z}}_{\mathbf{X}}(T) = \sum_{\vec{w} \in (\mathbb{Z}_N)^C} \Phi^B_{\mathbf{X}}(T, \vec{w})$$

where $(T, \vec{w})$ is a diagram of T with framing vector $\vec{w}$.

Example 2 The matrix

$$M_{\mathbf{X}} = \left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{array}\right]$$

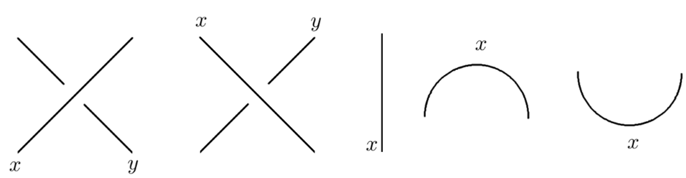
defines an involutory birack of rank $N = 2$. To compute the counting invariant $\Phi_{\mathbf{X}}^{\mathbb{Z}}(T)$ of a tangle T, we need to find the rack labelings over a complete set of framing vectors mod 2. For example, the Hopf link $L2a1$ has two components and thus the space of framings is $(\mathbb{Z}_n)2^2 = \{(0,0),(1,0),(0,1),(1,1)\}$. The Hopf link with framing vector $(0,1)$, for instance, has $\mathbf{X}$-labelings



so this framing contributes 3 labelings to the invariant. The other framing vectors contribute $1, 3$, and 5 labelings respectively, so we have $\phi_{\mathbf{X}}^{\mathbb{Z}}(L2a1) = 1 + 3 + 3 + 5 = 12$.

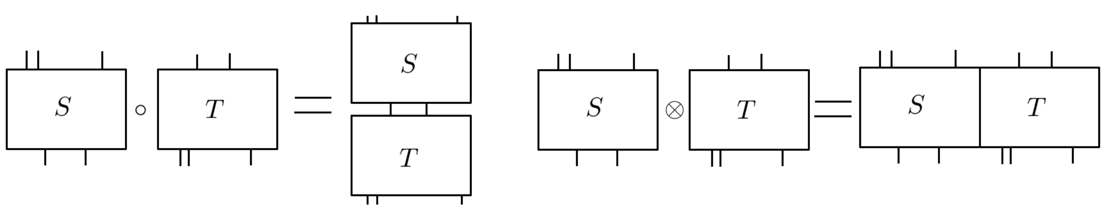
3 Quantum Enhancements

Let T be an unoriented link of c components and $\mathbf{X}$ a finite involutory birack of rank N. To each $\mathbf{X}$-labeling $f : IB(T) \to \mathbf{X}$ of T we would like to define a signature $\sigma(f)$ which is invariant under $\mathbf{X}$-labeled framed Reidemeister moves. We will do this by defining an $\mathbf{X}$-labeled tangle functor or $\mathbf{X}$-labeled quantum invariant which we call a *quantum weight* Q. More precisely, any $\mathbf{X}$-labeled unoriented tangle diagram T can, after applying planar isotopy if necessary, be divided into pieces of the pictured forms:

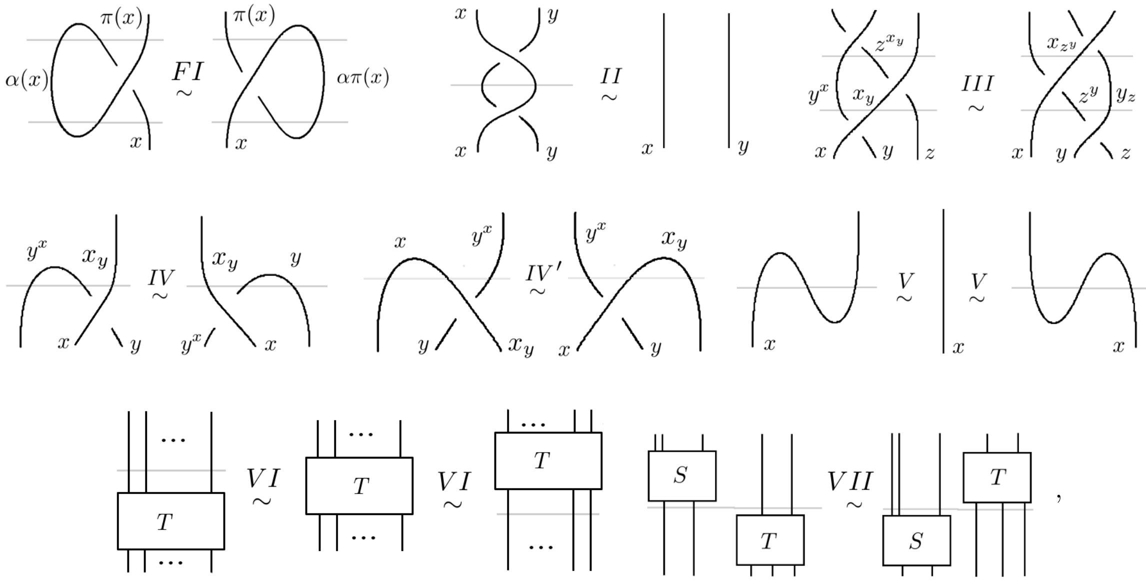


$X_{x,y}$ $\qquad$ $X^{-1}_{x,y}$ $\qquad$ I $\qquad$ N_x $\qquad$ U_x

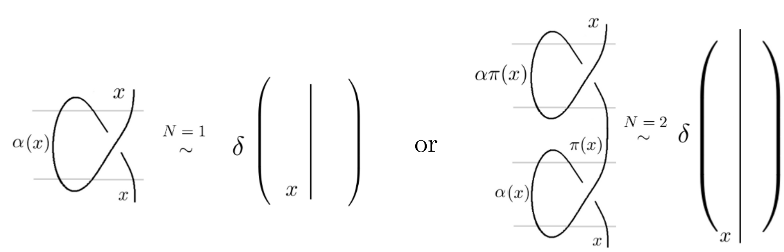
Now let us fix a field k and a k-vector space V. The idea is to assign linear maps to the basic tangles so that the overall tangle determines a linear map when we interpret horizontal stacking as tensor product and vertical stacking as composition of linear transformations.



A quantum weight will then be an assignment of linear transformations to these basic $\mathbf{X}$-labeled tangles such that equivalent $\mathbf{X}$-labeled framed tangles define the same linear transformation. This breaks down into the requirement that the assignment respects the **X**-*labeled framed tangle moves* (see [14, 8]):



together with the requirement that the appropriate **X**-labeled N-phone cord move acts as multiplication by a scalar δ.



Fixing a basis for V, we can regard sideways stacking as Krönecker product of matrices and vertical stacking as matrix product. The moves VI and VII are automatically satisfied if we choose the identity transformation for the I-tangle by the mixed-product property of Kronecker product. Thus we have:

Definition 2 Let $\mathbf{X}$ be an involutory birack with finite rank N, V a vector space over a field k and $I : V \to V$ the identity transformation on V. A *quantum weight* is an assignment of linear transformations

$$X_{x,y} : V \otimes V \to V \otimes V, \quad N_x : V \otimes V \to k, \quad U_x : k \to V \otimes V$$

indexed by $x, y \in \mathbf{X}$ satisfying the following conditions for all $x, y, z \in \mathbf{X}$:

(I) $(N_{\alpha(x)} \otimes I)(I \otimes X_{\alpha(x),x})(U_{\alpha(x)} \otimes I) = (I \otimes N_{\alpha\pi(x)})(X_{x,\alpha\pi(x)} \otimes I)(I \otimes U_{\alpha\pi(x)})$

(II) $X_{x,y}$ is invertible,

(III) $(X_{y^x,z^{x_y}} \otimes I)(I \otimes X_{x_y,z})(X_{x,y} \otimes I) = (I \otimes X_{x_{z^y},y_z})(X_{x,z^y} \otimes I)(I \otimes X_{y,z})$

(IV) $(N_{y^x} \otimes I)(I \otimes X_{x,y}) = (I \otimes N_y)(X^{-1}_{x_y,y} \otimes I)$

(IV′) $(N_x \otimes I)(I \otimes X^{-1}_{x,y^x}) = (I \otimes N_{x_y})(X_{x,y} \otimes I)$

(V) $(I \otimes N_x)(U_x \otimes I) = I = (N_x \otimes I)(I \otimes U_x)$

$$(VI) \begin{cases} (N_{\alpha(x)} \otimes I)(I \otimes X_{\alpha(x),x})(U_{\alpha(x)} \otimes I) = \delta I & N=1 \\ (N_{\alpha\pi(x)} \otimes I)(I \otimes X_{\alpha\pi(x),\pi(x)})(U_{\alpha\pi(x)} \otimes I)(N_{\alpha(x)} \otimes I)(I \otimes X_{\alpha(x),x})(U_{\alpha(x)} \otimes I) = \delta I & N=2 \end{cases}$$

for an invertible scalar δ.

Given an involutory birack $\mathbf{X} = \{x_1, \dots, x_m\}$ and a k-vector space V of dimension d with a fixed basis, we can specify a quantum weight with a quadruple (M, N, U, δ) where M is an $m \times m$ block matrix of $d^2 \times d^2$ blocks, with the (i, j) block of M the matrix with respect to a fixed choice of basis of the map $X_{x_i,x_j} : V \otimes V \to V \otimes V$ associated to the crossing with labels x_i and x_j, N is a block matrix of row vectors specifying the maps $N_x : V \otimes V \to k$, and U is a block matrix of column vectors specifying the maps $U_k : k \to V \otimes V$.

$$M = \left[\begin{array}{c|c|c|c} X_{11} & X_{12} & \dots & X_{1m} \\ \hline X_{21} & X_{22} & \dots & X_{2m} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline X_{m1} & X_{m2} & \dots & X_{mm} \end{array}\right], \quad N = \left[\begin{array}{c} N_1 \\ \hline N_2 \\ \hline \vdots \\ \hline N_m \end{array}\right], \quad \text{and} \quad U = \left[\begin{array}{c|c|c|c} U_1 & U_2 & \dots & U_m \end{array}\right].$$

Example 3 Let $B = \{1\}$, the singleton birack. Then the quadruple

$$(M, N, U, \delta) = \left(\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & 0 & A^{-1} & 0 \\ 0 & A^{-1} & A - A^{-3} & 0 \\ 0 & 0 & 0 & A \end{bmatrix}, \begin{bmatrix} 0 & A & -A^{-1} & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -A \\ A^{-1} \\ 0 \end{bmatrix}, -A^3 \right)$$

defines a well-known quantum weight, the Kauffman bracket/Jones polynomial. Indeed, for any involutory birack $\mathbf{X}$, we can set $X_{x,y}, N_x$ and U_x equal to these M, N and U matrices respectively for all $x, y \in \mathbf{X}$ and $\delta = -A^3$ to obtain a quantum weight.

Example 4 Let $\mathbf{X}$ be the constant action birack on $\{1, 2\}$ with $\sigma = \tau = (12)$; $\mathbf{X}$ has birack matrix

$$\left[\begin{array}{cc|cc} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right].$$

It is easy to verify that $\mathbf{X}$ is involutory (see [2]), and it is straightfoward (if somewhat tedious) to verify that the following quadruple defines a quantum weight of $\mathbf{X}$ where $V = \mathbb{Q}^2$:

$$\left(\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & b^{-1} & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & 0 & a^{-1} & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & a^{-1} & 0 \\ b^{-1} & 0 & 0 & 0 & b & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & b & 0 & 0 & 0 & b^{-1} \\ 0 & a^{-1} & 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & a^{-1} & 0 & 0 & 0 & a & 0 \\ b & 0 & 0 & 0 & b^{-1} & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cccc} 0 & n & -n & 0 \\ \hline 0 & -n & n & 0 \end{array}\right], \left[\begin{array}{c|c} 0 & 0 \\ -n^{-1} & n^{-1} \\ n^{-1} & -n^{-1} \\ 0 & 0 \end{array}\right], -a^{-1} \right).$$

A quantum weight Q translates an $\mathbf{X}$-labeling f of a tangle diagram T into a linear transformation $Q(f) : V^{\otimes n} \to V^{\otimes m}$ for some $n, m \in \mathbb{Z}$. In the special case when T is a closed tangle, i.e. a knot or link, $Q(f)$ is a scalar. Moreover, by construction $Q(f)$ is invariant under $\mathbf{X}$-labeled framed isotopy moves, and N-phone cord moves change $Q(f)$ by a power of δ. To obtain an unframed invariant, we correct for the effects of N-phone cord moves analogously to the normalization used to obtain the Jones polynomial from the Kauffman bracket: if the blackboard framing of a component c_k of T is w_k, write $w_k = q_k N + r_k$ where $0 \le r_k < N$, and define the *normalized quantum weight* $\overline{Q(f)}$ of an $\mathbf{X}$-labeled tangle diagram T by

$$\overline{Q(f)} = \delta^{-\vec{w}} Q(f) \qquad \text{where} \quad \delta^{-\vec{w}} = \prod_{k=1}^{c} \delta^{-q_k}.$$

Equivalently, the normalized quantum weight of a diagram is the quantum weight of the diagram equivalent to T by framed Reidemeister and N-phone cords moves whose framing is equal to its reduced value mod N. Then by construction we have

Theorem 2 *Let $\mathbf{X}$ be an involutory birack of rank N and Q a quantum weight. If T and T' are unoriented blackboard framed $\mathbf{X}$-labeled tangle diagrams which are related by framed isotopy and N-phone cord moves, then $\overline{Q(L)} = \overline{Q(L')}$.*

As a corollary, we can use quantum weightings to enhance the integral birack counting invariant. More precisely, we have

Definition 3 Let $\mathbf{X}$ be an involutory birack, V a k-vector space, Q a quantum weight and T an unoriented tangle of c components. The *quantum enhanced multiset invariant* of T is the multiset

$$\Phi_{\mathbf{X}}^{Q,M}(L) = \{\overline{Q(f)} \mid f \in \mathrm{Hom}(IB(T,\vec{w}),\mathbf{X}),\ \vec{w} \in \mathrm{Z}_N^c\}$$

and if T is a link, the *quantum enhanced polynomial invariant* of T with respect to $\mathbf{X}$ and Q is

$$\Phi_{\mathbf{X}}^{Q}(T) = \sum_{\vec{w}\in\mathbb{Z}_N^c} \left(\sum_{f\in\mathrm{Hom}(IB(T,\vec{w}),\mathbf{X})} u^{\overline{Q(f)}} \right).$$

In particular, we have

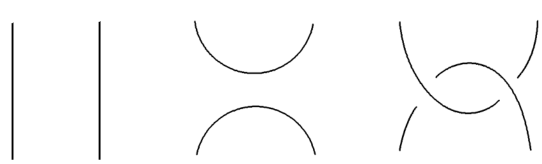
Theorem 3 *If $\mathbf{X}$ is an involutory birack, Q is a quantum weight and T and T' are ambient isotopic unoriented tangles, then $\Phi_{\mathbf{X}}^{Q,M}(T) = \Phi_{\mathbf{X}}^{Q,M}(T')$ and if T is a link, then $\Phi_{\mathbf{X}}^{Q}(T) = \Phi_{\mathbf{X}}^{Q}(T')$.*

Example 5 Let $\mathbf{X}$ be the birack with a single element. Then $\Phi_{\mathbf{X}}^{Q,M}$ is a tangle functor or quantum invariant, and indeed all such invariants can be regarded as special cases of $\Phi_{\mathbf{X}}^{Q,M}$ with $\mathbf{X} = \{1\}$. Alternatively, a quantum weight in which the maps $X_{x,y}, N_x, U_x$ do not depend on the birack labeling satisfies $\Phi_{\mathbf{X}}^{Q,M} = \Phi_{\mathbf{X}}^{\mathbb{Z}} \times Q$ where Q is a tangle functor.

Remark 1 Let $\mathbf{X}$ be a finite involutory birack of rank $N = 1$ with $B_2(x,y) = x$, also known as a *kei* (圭) or *involutory quandle*, and let $V = k$ be a one-dimensional vector space. Then a quantum weight assigns scalars $X_{x,y}$, N_x, and U_x to each pair of elements or element of $\mathbf{X}$ respectively, satisfying the above conditions. If $N_x = U_x = \delta = 1$ for all $x \in \mathbf{X}$, the function $\phi : \mathbf{X}\times\mathbf{X} \to k$ is then a quandle 2-cocycle and $\Phi_{\mathbf{X}}^{Q}(L)$ is the CJKLS quandle 2-cocycle invariant associated to ϕ. See [3] for more.

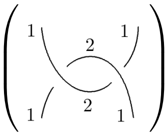
Remark 2 The anonymous referee of an earlier version of this paper suggested that an alternative way to view quantum enhancements would regard them as quantum invariants which admit gradings by biracks. Future work will no doubt explore this perspective.

Example 6 Let $\mathbf{X}$ be the constant action birack from example 4. Note that $\mathbf{X}$-labelings of a tangle simply switch the label from 1 to 2 or 2 to 1 at every overcrossing and undercrossing point, and moreover $\Phi_{\mathbf{X}}^{\mathbb{Z}}(T) = 2^c$ where c is the number of components of T. Thus, $\Phi_{\mathbf{X}}^{\mathbb{Z}}$ does not distinguish any pair of tangles with the same number of components. However, consider the following two-component tangles, each with $\Phi_{\mathbf{X}}^{\mathbb{Z}}(T) = 4$:



T_1 T_2 T_3

The **X**-labeling of T_3 below, for instance, contributes

$\sigma\Big($  $\Big)$

$$
\begin{aligned}
&= (I\otimes N_2\otimes I)(X_{1,2}^{-1}\otimes X_{2,1}^{-1})(I\otimes U_2\otimes I)\\
&= \left(\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\otimes\begin{bmatrix} 0 & -n & n & 0\end{bmatrix}\otimes\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\right)\\
&\quad\left(\begin{bmatrix} 0 & 0 & 0 & b^{-1}\\ 0 & a & 0 & 0\\ 0 & 0 & a & 0\\ b^{-1} & 0 & 0 & 0\end{bmatrix}\otimes\begin{bmatrix} 0 & 0 & 0 & b^{-1}\\ 0 & a & 0 & 0\\ 0 & 0 & a & 0\\ b^{-1} & 0 & 0 & 0\end{bmatrix}\right)\left(\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\otimes\begin{bmatrix} 0\\ n^{-1}\\ -n^{-1}\\ 0\end{bmatrix}\otimes\begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\right)\\
&= \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & a^{-2} & -b^2 & 0\\ 0 & -b^2 & a^{-2} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}
\end{aligned}
$$

to $\Phi_{\mathbf{X}}^{Q,M}(T_2)$ since $\vec{w}=(0,0)$ so $\overline{Q(f)}=Q(f)$; repeating for the other **X**-labelings and for the other tangles yields the invariant values listed below.

$$
\begin{aligned}
\Phi_{\mathbf{X}}^{Q,M}(T_1) &= \left\{4\times\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\right\},\\
\Phi_{\mathbf{X}}^{Q,M}(T_2) &= \left\{2\times\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, 2\times\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\right\},\\
\Phi_{\mathbf{X}}^{Q,M}(T_2) &= \left\{2\times\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & a^{-2} & -b^2 & 0\\ 0 & -b^2 & a^{-2} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, 2\times\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & -a^2 & b^{-2} & 0\\ 0 & b^{-2} & -a^2 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\right\}.
\end{aligned}
$$

Note that the invariant $\Phi_{\mathbf{X}}^{Q}$ determines the integral birack counting invariant since $|\Phi_{\mathbf{X}}^{M,Q}|=\Phi_{\mathbf{X}}^{\mathbb{Z}}$ and, if T is a link, evaluating $\Phi_{\mathbf{X}}^{Q}$ at $u=1$ yields $\Phi_{\mathbf{X}}^{\mathbb{Z}}$. As example 4 shows, $\Phi_{\mathbf{X}}^{Q}$ is a stronger invariant in general than $\Phi_{\mathbf{X}}^{\mathbb{Z}}$ and thus a proper enhancement.

4 Quantum Enhancements of X-labeled Braids

Every framed oriented knot or link can be expressed as a closed braid $\hat{\beta}$ for a braid $\beta=\sigma_{k_1}\dots\sigma_{k_n}$. Restricting our attention to closed braids yields provides advantages for searching for quantum weights: we need only seek matrices $\sigma_{a,b}^{\pm 1}$ satisfying the Reidemeister III move, and we can consider matrices with arbitrary dimension instead of being limited to perfect square dimensions as in the tangle case. On the other hand, any pair of conjugate closed braids determine the same knot, so the weight matrix determined by an **X**-labeled closed braid is not a valid signature, only its conjugacy class. We will deal with this by taking the trace of the weight matrix as the signature, which is unchanged by conjugacy as well as faster computationally than other conjugacy-class invariants such as the determinant.

Definition 4 Let B_n be the n-strand braid group and let **X** be an involutory birack. An *n-braid weight* for **X** is an assignment of an invertible matrix $\sigma_j^{x,y}$ with entries in a ring R for each $x,y\in\mathbf{X}$ and each

$j = 1, \ldots, n-1$ such that the $\mathbf{X}$-labeled braid relations

$$\sigma_j^{x,y} \sigma_{j+1}^{x_y,z} \sigma_j^{y^x,z^{x_y}} = \sigma_{j+1}^{y,z} \sigma_j^{x,z^y} \sigma_{j+1}^{x_{z^y},y_z}$$

and

$$\sigma_j^{x,y} \sigma_k^{u,v} = \sigma_k^{u,v} \sigma_j^{x,y} \quad |j-k| < 2$$

are satisfied.

Given a braid weight for an involutory birack X, we can define an enhancement of the involutory birack counting invariant for closed n-braids by replacing each braid generator in an $\mathbf{X}$-labeling of $\hat{\beta}$ with the corresponding matrix and taking the trace as a signature. Note that an $\mathbf{X}$-labeling of a closed braid must have the same list of $\mathbf{X}$-labels along the top and bottom of the braid in addition to satisfying the crossing condition at each crossing, and note further that fixing the braid index fixes the framing, so in this section we are only enhancing $\Phi_{\mathbb{X}}^B$.

Definition 5 Let $\beta \in B_n$, $\mathbf{X}$ be an involutory birack, and $W = \{\sigma_j^{x,y} \mid x, y \in X, 1 \leq j \leq n-1\}$ be a braid weight for X. Then the *braid weight enhancement of the basic* $\mathbf{X}$*-counting invariant* is the multiset of the traces of the matrices β_f obtained by replacing each braid group generator σ_j with the appropriate matrix $X_j^{x,y}$ over the set of all $\mathbf{X}$-labelings f of β, i.e.

$$\Phi_{\mathbf{X}}^{M,W}(\beta) = \{\mathrm{tr}\ (\beta_W) \mid f \in \mathrm{Hom}(IB(\beta), \mathbf{X})\}$$

with polynomial version

$$\Phi_{\mathbf{X}}^{W}(\beta) = \sum_{f \in \mathrm{Hom}(IB(\beta), \mathbf{X})} u^{\mathrm{tr}\ (\beta_W)}.$$

Example 7 Let $\mathbf{X}$ be the involutory birack with matrix

$$M_{\mathbf{X}} = \left[\begin{array}{cc|cc} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right].$$

Our computer search identified 3-braid weights for $\mathbf{X}$ including

$j=1$		$j=2$	
$\begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ y & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ w & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ z & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & 1 \\ z & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ w & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ y & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix}$

Then for instance the closure of the braid $\beta = \sigma_1\sigma_1\sigma_1\sigma_2 \in B_3$ is the trefoil knot with framing number 2; it has $\Phi_{\mathbf{X}}^{M,W}(\beta) = \{2y^2, 2z^2\}$ while the closure of the braid $\beta' = \sigma_1\sigma_1^{-1}\sigma_1\sigma_2$ is an unknot with $\Phi_{\mathbf{X}}^{M,W}(\beta) = \{2y, 2z\}$.

5 Questions

We conclude with a few questions and directions for future research.

Our computations suggest that the quantum enhancement in example 4 is trivial on classical knots, and indeed many of the two-dimensional quantum enhancements we were able to find in our (far from exhaustive) computer search with respect to small-cardinality involutory biracks $\mathbf{X}$ appear to yield trivial invariants on closed classical knots. Nevertheless, the enhancement in example 4 is nontrivial on at least some tangles with nonempty boundary, and there are many known examples of nontrivial one-dimensional quantum weights (namely, CJKLS quandle 2-cocycle invariants) and nontrivial two-dimensional quantum weights on small cardinality biracks (namely, tangle functor invariants).

Expanding on example 5, let us say a quantum weight is *homogeneous* if $X_{x,y} = X_{x',y'}$, $N_x = N_{x'}$ and $U_x = U_{x'}$ for all $x, y, x', y' \in \mathbf{X}$, and that a quantum weight is *heterogeneous* if at least one $X_{x,y} \neq X_{x'y'}$, $N_x \neq N_{x'}$ or $U_x \neq U_{x'}$. Example 4 demonstrates the existence of heterogeneous quantum enhancements. For a given involutory birack $\mathbf{X}$, what is the minimal dimension of a vector space V required for the existence of a heterogeneous quantum weight?

Continuing in the same vein, say a quantum weight is *strongly heterogeneous* if at least one of the $X_{i,j}$ matrices is not a classical R-matrix, i.e. if $X_{i,j}$ does not satisfy the unlabeled Yang-Baxter equation

$$(I \times X_{i,j})(X_{i,j} \times I)(I \times X_{i,j}) = (X_{i,j} \times I)(I \times X_{i,j})(X_{i,j} \times I).$$

If $\mathbf{X}$ is an involutory quandle, then the matrices X_{ii} on the diagonal of X must be R-matrices, but even in this case the off-diagonal matrices need not satisfy the unlabeled Yang-Baxter equation *a priori*, and the quantum weights associated to a non-quandle birack might all be fail to be classical R-matrices. Such quantum weights are expected to define the most interesting and non-trivial quantum enhancements.

We note that in general, finding quantum weights is a difficult problem. Even for the smallest non-trivial biracks (those with two elements) and the smallest non-scalar quantum weights ($\dim(V) = 2$), the entries of the matrices represent up to 80 independent variables with the axioms yielding a system of hundreds of non-linear equations. Computer searches have yielded some results, but better methods of finding quantum weights would be of great interest. Our `python` code is available at `www.esotericka.org`.

Finally, we note that in this paper we have considered only the simplest possible case, that of unoriented classical tangles. In future papers we will explore the oriented, virtual and twisted virtual case, each of which involves more complicated axioms and, we expect, richer structure.

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