Quantum Enhancements and Biquandle Brackets

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Abstract

We introduce a new class of quantum enhancements we call *biquandle brackets*, which are customized skein invariants for biquandle colored links. Quantum enhancements of biquandle counting invariants form a class of knot and link invariants that includes biquandle cocycle invariants and skein invariants such as the HOMFLY-PT polynomial as special cases, providing an explicit unification of these apparently unrelated types of invariants. We provide examples demonstrating that the new invariants are not determined by the biquandle counting invariant, the knot quandle, the knot group or the traditional skein invariants.

KEYWORDS: biquandles, biquandle brackets, quantum invariants, quantum enhancements of counting invariants

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1 Introduction

Biquandles, algebraic structures with axioms derived from the Reidemeister moves for oriented knots, were introduced in [6] and have been used to define invariants of oriented knots and links in [5, 10] and more. In particular, the number of biquandle colorings of an oriented knot or link diagram K by a finite biquandle X defines a nonnegative integer-valued invariant known as the *biquandle counting invariant*, denoted $\Phi_X^{\mathbb{Z}}(K)$. An *enhancement* of $\Phi_X^{\mathbb{Z}}$ is a generally stronger invariant from which $\Phi_X^{\mathbb{Z}}$ can be recovered; enhancements have been studied in [2, 3, 8, 13] to name just a few.

In [12] the first and last listed authors introduced the notion of *quantum enhancements* of $\Phi_X^{\mathbb{Z}}$ defined as quantum invariants of biquandle-colored knot or link diagrams, focusing on the unoriented case. In this paper we introduce a new infinite family of quantum enhancements using *biquandle brackets*, i.e., skein relations which depend on biquandle colorings. This family of invariants includes biquandle counting invariants, biquandle (and quandle) cocycle invariants, and classical quantum invariants such as the Jones and HOMFLYPT polynomials (see for example [11]) as special cases. In particular, we provide examples of *strongly heterogeneous* quantum enhancements, i.e., solutions to the biquandle-colored Yang-Baxter equation which are not solutions to the uncolored Yang-Baxter equation, settling a question from [12] and confirming that there are quantum enhancements which are neither cocycle invariants nor classical skein invariants.

The biquandle bracket conditions we find are very similar to the biquandle 2-cocycle condition, and indeed biquandle 2-cocycle invariants form a special case of biquandle brackets. Moreover, we identify an equivalence relation on biquandle brackets yielding the same invariant which specializes to the cohomology relation for biquandle cocycles, even for non-cocycle biquandle brackets. Connections between quantum invariants and quandle cocycle invariants were also studied in [7].

The paper is organized as follows. In Section 2 we review the basics of biquandles and the biquandle counting invariant. In Section 3 we define biquandle brackets and provide some examples, including as an application a new skein invariant with values in the Galois field of eight elements $\mathbb{F}_8$. In Section 4 we consider the special case of biquandle brackets when X is a quandle. We end in Section 5 with some open questions for future research.

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2 Biquandles

A *biquandle* is a set X with two binary operations $\underline{\triangleright}, \overline{\triangleright} : X \times X \to X$ satisfying for all $x, y, z \in X$

(i) $x \,\underline{\triangleright}\, x = x \,\overline{\triangleright}\, x$,

(ii) the maps $\alpha_y, \beta_y : X \to X$ and $S : X \times X \to X \times X$ defined by $\alpha_y(x) = x \,\overline{\triangleright}\, y$, $\beta_y(x) = x \,\underline{\triangleright}\, y$ and $S(x, y) = (y \,\overline{\triangleright}\, x, x \,\underline{\triangleright}\, y)$ are invertible, and

(iii) the *exchange laws* are satisfied:

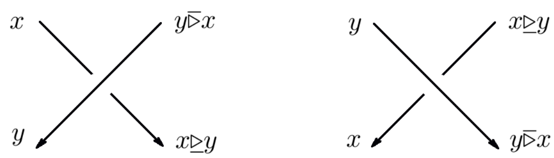
$$\begin{aligned}
(x \,\underline{\triangleright}\, y) \,\underline{\triangleright}\, (z \,\underline{\triangleright}\, y) &= (x \,\underline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z) \\
(x \,\underline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\underline{\triangleright}\, y) &= (x \,\overline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z) \\
(x \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\overline{\triangleright}\, y) &= (x \,\overline{\triangleright}\, z) \,\overline{\triangleright}\, (y \,\underline{\triangleright}\, z).
\end{aligned}$$

If $x \,\overline{\triangleright}\, y = x$ for all $x, y \in X$, we say X is a *quandle*.

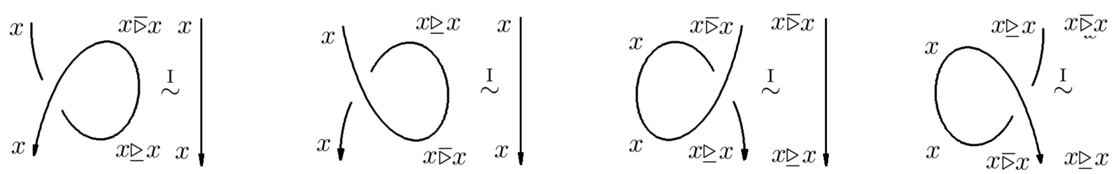
If X and Y are biquandles, then a *biquandle homomorphism* is a map $f : X \to Y$ such that for all $x, y \in X$, we have

$$f(x \,\underline{\triangleright}\, y) = f(x) \,\underline{\triangleright}\, f(y) \quad \text{and} \quad f(x \,\overline{\triangleright}\, y) = f(x) \,\overline{\triangleright}\, f(y).$$

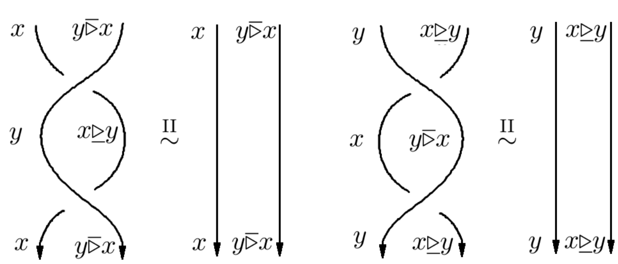
The biquandle axioms come from the *Reidemeister moves* where we interpret $x \,\underline{\triangleright}\, y$ as x crossing under y and $y \,\overline{\triangleright}\, x$ as y crossing over x from left to right when the crossing has both strands oriented down as shown.



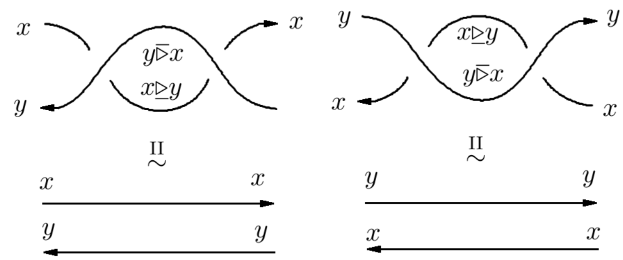
Then the biquandle axioms are the conditions required for every valid biquandle coloring of the semiarcs in a knot diagram before a move to correspond to a unique valid biquandle coloring (i.e., coloring satisfying the condition pictured above at every crossing) of the diagram after the move. All four oriented type I moves require that $x \,\underline{\triangleright}\, x = x \,\overline{\triangleright}\, x$.



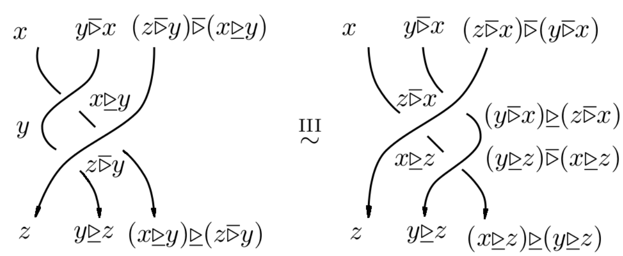
The *direct* type II moves, in which the strands are oriented in the same direction, require that $y \,\overline{\triangleright}\, x$ and $x \,\underline{\triangleright}\, y$ are right-invertible.



The reverse type II moves, in which the strands are oriented in opposite directions, require the map $(x, y) \mapsto (y \,\overline{\triangleright}\, x, x \,\underline{\triangleright}\, y)$ to be invertible.



Finally, the exchange laws result from the Reidemeister III move.



Example 1. Let X be any set and $\sigma : X \to X$ any bijection. Then X is a biquandle with operations

$$x \,\underline{\triangleright}\, y = x \,\overline{\triangleright}\, y = \sigma(x)$$

known as a *constant action biquandle*. If σ is the identity, then X is a *trivial quandle*.

Example 2. Let $\ddot{\Lambda} = \mathbb{Z}[t^{\pm 1}, r^{\pm 1}]$. Then any $\ddot{\Lambda}$-module A is a biquandle, known as an *Alexander biquandle*, under the operations

$$x \,\underline{\triangleright}\, y = tx + (r^{-1} - t)y \quad \text{and} \quad x \,\overline{\triangleright}\, y = r^{-1}y.$$

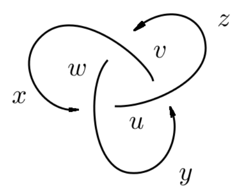
In particular, any commutative ring A becomes an Alexander biquandle with a choice of invertible elements $t, r \in A$.

We can express the biquandle operations on a set $X = \{x_1, \dots, x_n\}$ with operation tables for $\underline{\triangleright}$ and $\overline{\triangleright}$ expressed as an $n \times 2n$ block matrix such that the entries in row k columns j and $n + j$ are the subscripts of $x_k \,\underline{\triangleright}\, x_j$ and $x_k \,\overline{\triangleright}\, x_j$ respectively.

Example 3. The Alexander biquandle structure on $\mathbb{Z}_5 = \{1, 2, 3, 4, 5\}$ (where 5 represents the class of 0 so our block rows and columns are numbered 1 through 5) with $t = 2$ and $r = 4$ can be expressed as the block matrix

$$\left[\begin{array}{ccccc|ccccc}
4 & 1 & 3 & 5 & 2 & 4 & 4 & 4 & 4 & 4 \\
1 & 3 & 5 & 3 & 4 & 3 & 3 & 3 & 3 & 3 \\
3 & 5 & 2 & 4 & 1 & 2 & 2 & 2 & 2 & 2 \\
5 & 2 & 4 & 1 & 3 & 1 & 1 & 1 & 1 & 1 \\
2 & 4 & 1 & 3 & 5 & 5 & 5 & 5 & 5 & 5
\end{array}\right].$$

Example 4. Let L be a tame oriented knot or link. The *fundamental biquandle* of L, denoted $\mathcal{B}(L)$, is the set of equivalence classes of biquandle words in a set of generators corresponding with the semiarcs in a diagram of L under the equivalence relation generated by the crossing relations of L and the biquandle axioms. For instance, the trefoil knot 3_1



has the fundamental biquandle presentation

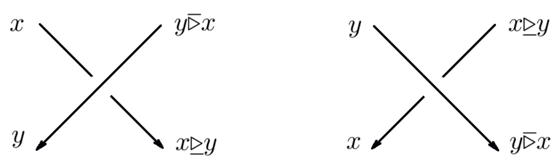
$$\mathcal{B}(3_1) = \langle x, y, z, u, v, w \mid x \,\underline{\triangleright}\, y = u, y \,\overline{\triangleright}\, x = w, y \,\underline{\triangleright}\, z = v, z \,\overline{\triangleright}\, y = u, z \,\underline{\triangleright}\, x = w, x \,\overline{\triangleright}\, z = v \rangle.$$

Then for instance in $\mathcal{B}(3_1)$ we have

$$(y \,\overline{\triangleright}\, u) \,\overline{\triangleright}\, (x \,\overline{\triangleright}\, u) = (y \,\overline{\triangleright}\, x) \,\overline{\triangleright}\, (u \,\underline{\triangleright}\, x) = w \,\overline{\triangleright}\, (u \,\underline{\triangleright}\, x).$$

Different diagrams of the same knot or link yield different presentations which differ by Tietze moves and hence present the same biquandle.

Given a finite biquandle X and a tame knot or link diagram L, a *biquandle coloring* of L is an assignment of elements of X to the semiarcs in L such that the crossing relations

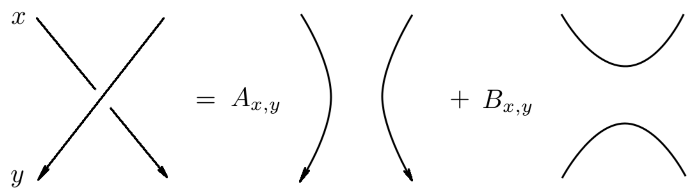


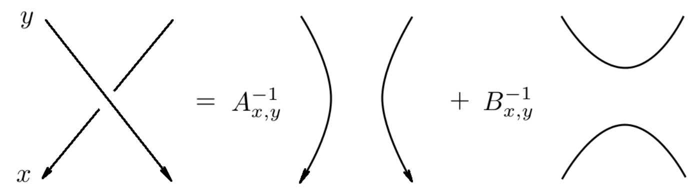
are satisfied at every crossing. Such an assignment determines and is determined by a biquandle homomorphism $f : \mathcal{B}(L) \to X$. In particular, the set of biquandle colorings of L can be identified with the set $\mathrm{Hom}(\mathcal{B}(L), X)$ of biquandle homomorphisms from the fundamental biquandle of L to X. If L is tame, then $\mathcal{B}$ is finitely generated with $2n$ generators where n is the number of semiarcs in L; hence $|\mathrm{Hom}(\mathcal{B}(L), X)| \leq |X|^{2n}$. We usually write $|\mathrm{Hom}(\mathcal{B}(L), X)| = \Phi_X^{\mathbb{Z}}(L) \in \mathbb{N}$; this cardinality is known as the *biquandle counting invariant* [2].

Example 5. The trefoil knot 3_1 from example 4 has only one valid biquandle coloring by the Alexander biquandle in example 3, the labeling with every semiarc labeled 5, as can be determined by row-reducing over $\mathbb{Z}_5$ the coefficient matrix of the system of crossing equations or by brute-force checking all possible colorings and counting those which satisfy the crossing relations.

3 Biquandle Brackets

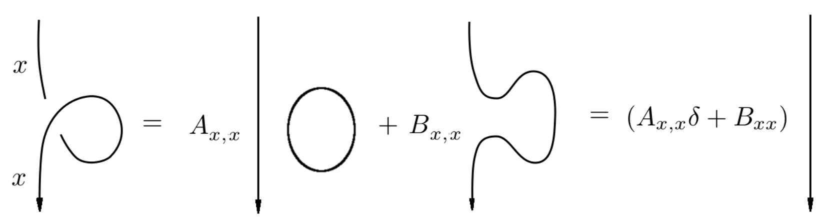
We would like to define a skein invariant (see [11] for instance) for biquandle-labeled link diagrams. Let X be a finite biquandle, and let us fix a commutative ring with identity R and denote the set of units of R as $R^{\times}$. We would like to choose elements $A_{x,y}, B_{x,y}, w \in R^{\times}$ and $\delta \in R$ such that the element of R determined by the skein relations



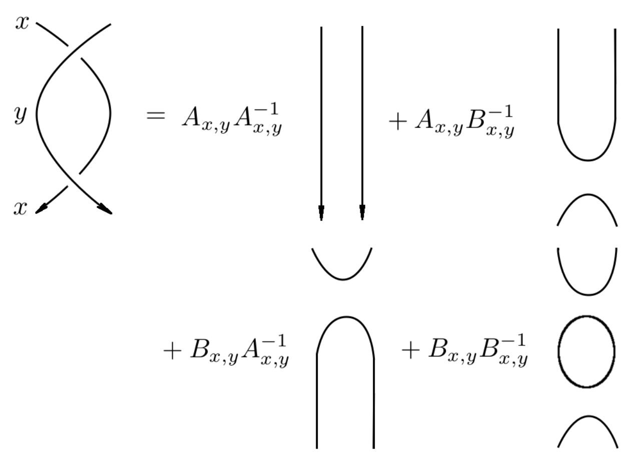


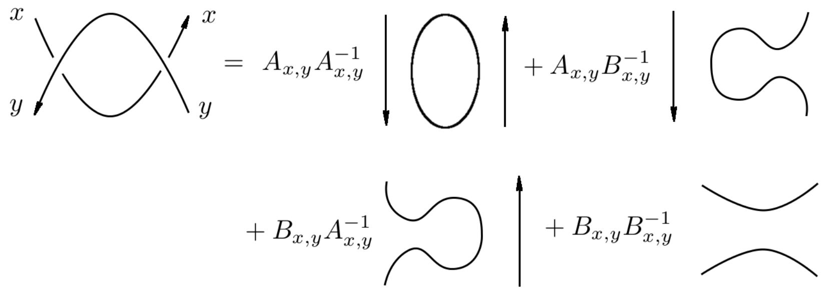
with δ the value of a simple closed curve and w the value of a positive kink is an invariant of X-labeled Reidemeister moves.

The first Reidemeister move comes in four oriented versions; the two positively oriented moves require that for all $x \in X$, we have $A_{x,x}\delta + B_{x,x} = w$, while the negatively oriented moves require $A_{x,x}^{-1}\delta + B_{x,x}^{-1} = w^{-1}$. In particular, we can think of writhe-reducing type I moves as factoring out powers of w and writhe-increasing type I moves as factoring out powers of w^{-1}.

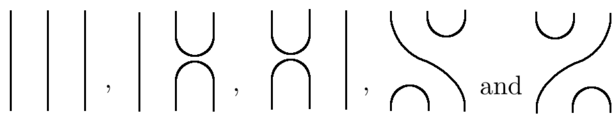


The direct type II moves require the oriented smoothing coefficients at positive and negative crossings to be multiplicative inverses, with the reverse II moves requiring the same of the unoriented smoothing coefficients; both moves then require that $\delta = -A_{x,y}B_{x,y}^{-1} - A_{x,y}^{-1}B_{x,y}$.

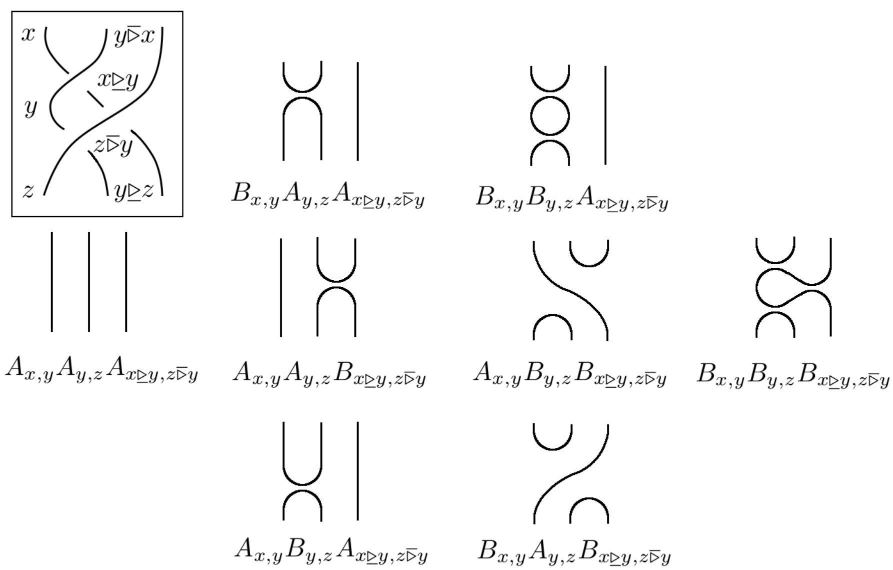




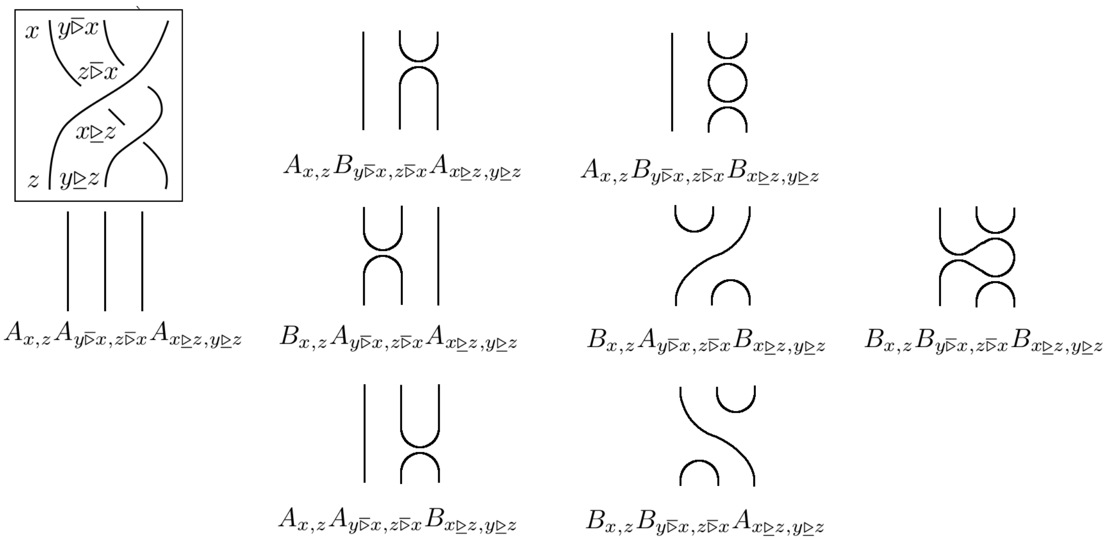
Comparing coefficients of the five crossingless diagrams



on both sides of the X-labeled Reidemeister III move, we have on the left side



and on the right side



yielding the remaining conditions on $A_{x,y}$ and $B_{x,y}$. We thus obtain

Definition 1. Let X be a finite biquandle and R be a commutative ring with identity. A *biquandle bracket* on X with values in R, also called an X*-bracket*, is a pair of maps $A, B : X \times X \to R^\times$ and distinguished elements $\delta \in R$ and $w \in R^\times$ satisfying

(i) for all $x \in X$,
$$\delta A_{x,x} + B_{x,x} = w \quad \text{and} \quad \delta A_{x,x}^{-1} + B_{x,x}^{-1} = w^{-1}$$

(ii) for all $x, y \in X$,
$$\delta = -A_{x,y}B_{x,y}^{-1} - A_{x,y}^{-1}B_{x,y}$$
and

(iii) for all $x, y, z \in X$,
$$\begin{array}{rcl}
A_{x,y}A_{y,z}A_{x\underline{\triangleright}y,z\overline{\triangleright}y} & = & A_{x,z}A_{y\overline{\triangleright}x,z\overline{\triangleright}x}A_{x\underline{\triangleright}z,y\underline{\triangleright}z} \\
A_{x,y}B_{y,z}B_{x\underline{\triangleright}y,z\overline{\triangleright}y} & = & B_{x,z}B_{y\overline{\triangleright}x,z\overline{\triangleright}x}A_{x\underline{\triangleright}z,y\underline{\triangleright}z} \\
B_{x,y}A_{y,z}B_{x\underline{\triangleright}y,z\overline{\triangleright}y} & = & B_{x,z}A_{y\overline{\triangleright}x,z\overline{\triangleright}x}B_{x\underline{\triangleright}z,y\underline{\triangleright}z} \\
A_{x,y}A_{y,z}B_{x\underline{\triangleright}y,z\overline{\triangleright}y} & = & A_{x,z}B_{y\overline{\triangleright}x,z\overline{\triangleright}x}A_{x\underline{\triangleright}z,y\underline{\triangleright}z} + A_{x,z}A_{y\overline{\triangleright}x,z\overline{\triangleright}x}B_{x\underline{\triangleright}z,y\underline{\triangleright}z} \\
 & & +\delta A_{x,z}B_{y\overline{\triangleright}x,z\overline{\triangleright}x}B_{x\underline{\triangleright}z,y\underline{\triangleright}z} + B_{x,z}B_{y\overline{\triangleright}x,z\overline{\triangleright}x}B_{x\underline{\triangleright}z,y\underline{\triangleright}z} \\
B_{x,y}A_{y,z}A_{x\underline{\triangleright}y,z\overline{\triangleright}y} + A_{x,y}B_{y,z}A_{x\underline{\triangleright}y,z\overline{\triangleright}y} & & \\
+\delta B_{x,y}B_{y,z}A_{x\underline{\triangleright}y,z\overline{\triangleright}y} + B_{x,y}B_{y,z}B_{x\underline{\triangleright}y,z\overline{\triangleright}y} & = & B_{x,z}A_{y\overline{\triangleright}x,z\overline{\triangleright}x}A_{x\underline{\triangleright}z,y\underline{\triangleright}z}
\end{array}$$

where $A(x,y)$ and $B(x,y)$ are denoted $A_{x,y}$ and $B_{x,y}$.

Given a finite biquandle $X = \{x_1, \dots, x_n\}$, an X-bracket can be represented by a pair of $n \times n$ matrices A, B with $A_{j,k} = A(x_j, x_k)$ and $B_{j,k} = B(x_j, x_k)$. We will usually write these as a single $n \times 2n$ block matrix for convenience. Note that we can recover δ and w from such a matrix, with
$$\delta = -A_{1,1}B_{1,1}^{-1} - A_{1,1}^{-1}B_{1,1} \quad \text{and} \quad w = A_{1,1}\delta + B_{1,1}.$$

Example 6. Let $X = \{1\}$ be the biquandle with one element. Then the matrix

$$\left[\ A \mid A^{-1}\ \right]$$

where $A \in \mathbb{Z}[A^{\pm 1}]$ is an invertible variable defines a biquandle bracket with

$$\delta = -A(A^{-1})^{-1} - (A^{-1})A^{-1} = -A^2 - A^{-2} \quad \text{and} \quad w = A(-A^2 - A^{-2}) + A^{-1} = -A^3.$$

Indeed, this is the Kauffman bracket (see for example [11, 14]).

Example 7. Let X be a finite biquandle, R be a commutative ring, and $C : X \to R^{\times}$ be a map where we write C_x for $C(x)$. Then the maps $A, B : X \times X \to R^{\times}$ defined by

$$A(x,y) = B(x,y) = C_x C_y^{-1} C_{x \,\underline{\triangleright}\, y}^{-1} C_{y \,\overline{\triangleright}\, x}$$

for all $x, y \in X$ define a biquandle bracket with $\delta = -2$ and $w = -1$. To see this, we note that if $A_{x,y} = B_{x,y}$, we necessarily have $\delta = -2$, and biquandle bracket axiom (iii)'s five equations all reduce to the first equation, namely

$$A_{x,y} A_{y,z} A_{x \,\underline{\triangleright}\, y, z \,\overline{\triangleright}\, y} = A_{x,z} A_{y \,\overline{\triangleright}\, x, z \,\overline{\triangleright}\, x} A_{x \,\underline{\triangleright}\, z, y \,\underline{\triangleright}\, z}.$$

Then

$$\begin{aligned} A_{x,y} A_{y,z} A_{x \,\underline{\triangleright}\, y, z \,\overline{\triangleright}\, y} &= (C_x C_y^{-1} C_{x \,\underline{\triangleright}\, y}^{-1} C_{y \,\overline{\triangleright}\, x})(C_y C_z^{-1} C_{y \,\underline{\triangleright}\, z}^{-1} C_{z \,\overline{\triangleright}\, y})(C_{x \,\underline{\triangleright}\, y} C_{z \,\overline{\triangleright}\, y}^{-1} C_{(x \,\underline{\triangleright}\, y) \,\underline{\triangleright}\, (z \,\overline{\triangleright}\, y)}^{-1} C_{(z \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (x \,\underline{\triangleright}\, y)}) \\ &= C_x C_{y \,\overline{\triangleright}\, x} C_z^{-1} C_{y \,\underline{\triangleright}\, z}^{-1} C_{(x \,\underline{\triangleright}\, y) \,\underline{\triangleright}\, (z \,\overline{\triangleright}\, y)}^{-1} C_{(z \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (x \,\underline{\triangleright}\, y)} \end{aligned}$$

while

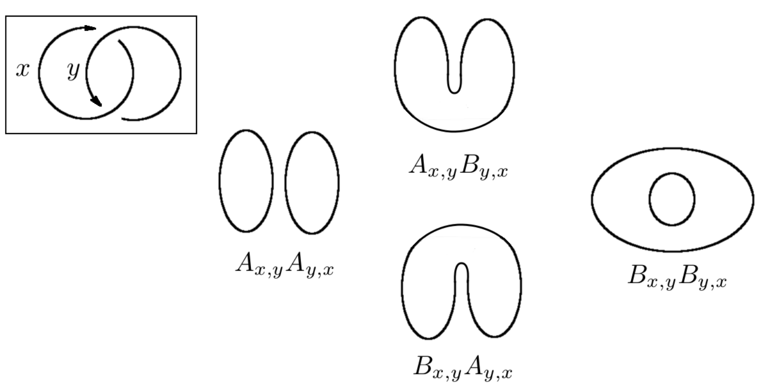
$$\begin{aligned} A_{x,z} A_{y \,\overline{\triangleright}\, x, z \,\overline{\triangleright}\, x} A_{x \,\underline{\triangleright}\, z, y \,\underline{\triangleright}\, z} &= (C_x C_z^{-1} C_{x \,\underline{\triangleright}\, z}^{-1} C_{z \,\overline{\triangleright}\, x})(C_{y \,\overline{\triangleright}\, x} C_{z \,\overline{\triangleright}\, x}^{-1} C_{(y \,\overline{\triangleright}\, x) \,\underline{\triangleright}\, (z \,\overline{\triangleright}\, x)}^{-1} C_{(z \,\overline{\triangleright}\, x) \,\overline{\triangleright}\, (y \,\overline{\triangleright}\, x)}) C_{x \,\underline{\triangleright}\, z}) \\ &\quad \times (C_{y \,\underline{\triangleright}\, z}^{-1} C_{(x \,\underline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\underline{\triangleright}\, z)}^{-1} C_{(y \,\underline{\triangleright}\, z) \,\overline{\triangleright}\, (x \,\underline{\triangleright}\, z)}) \\ &= C_x C_z^{-1} C_{y \,\overline{\triangleright}\, x} C_{(z \,\overline{\triangleright}\, x) \,\overline{\triangleright}\, (y \,\overline{\triangleright}\, x)} C_{y \,\underline{\triangleright}\, z}^{-1} C_{(x \,\underline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\underline{\triangleright}\, z)}^{-1} \end{aligned}$$

which are equal by the exchange laws.

We now introduce the first of our new invariants.

Definition 2. Let L be an oriented knot or link diagram with n crossings with generators $x_1, \dots, x_{2n}$ for the fundamental biquandle $\mathcal{B}(L)$ associated to the semiarcs. There are 2^n *states* corresponding to choices of oriented or unoriented smoothing for each crossing, each of which has an associated product of n factors of $A_{x,y}^{\pm 1}$ or $B_{x,y}^{\pm 1}$ times δ^k where k is the number of components of the state. The sum of these contributions times the *writhe correction factor*, w^{n-p}, is the *fundamental biquandle bracket value* for L.

Example 8. The Hopf link $L2a1$ below has four smoothed states with coefficients as listed.



Then the Hopf link has fundamental biquandle bracket value

$$\phi = w^{-2}(A_{x,y}A_{y,x}\delta^2 + B_{x,y}A_{y,x}\delta + A_{x,y}B_{y,x}\delta + B_{x,y}B_{y,x}\delta^2)$$

where x, y are generators of the fundamental biquandle $\mathcal{B}(L2a1) = \langle x, y \mid x \,\underline{\triangleright}\, y = x \,\overline{\triangleright}\, y,\ y \,\underline{\triangleright}\, x = y \,\overline{\triangleright}\, x\rangle$.

The fundamental biquandle bracket treats every knot or link as colored by elements of its fundamental biquandle. This fundamental biquandle bracket may be a complete invariant of virtual links since it includes the fundamental biquandle, already conjectured to be a complete invariant for virtual links up to a type of reflection [9], and our later examples demonstrate that the fundamental biquandle bracket can detect mirror images. However, comparing fundamental biquandle bracket values for different knots and link is not straightforward since any two such links are being colored by generally different biquandles.

To get a more immediately useful invariant, let X be a finite biquandle. For any X-bracket β over R, evaluating the fundamental biquandle bracket value of an X-coloring f of an oriented link diagram L yields an element of R which is unchanged by X-colored Reidemeister moves on L; let us denote this value by $\beta(f)$.

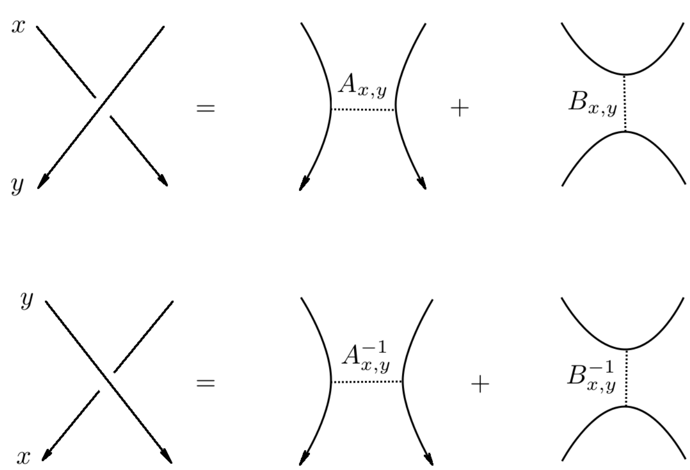
Definition 3. Let X be a finite biquandle, L an oriented link and β a biquandle bracket. Then the *biquandle bracket multiset* invariant of L is the multiset of β-values over the set of X-labelings of L,

$$\Phi_X^{\beta,M}(L) = \{\beta(f) \mid f \in \mathrm{Hom}(\mathcal{B}(L), X)\}$$

and the *biquandle bracket polynomial* invariant of L is

$$\Phi_X^{\beta}(L) = \sum_{f\in\mathrm{Hom}(\mathcal{B}(L),X)} u^{\beta(f)}.$$

Remark 1. Skein invariants of uncolored diagrams are often computed by expanding crossings one at a time rather than using the "state-sum" method of collecting all smoothings simultaneously [11]; for this method, we note that the smoothing coefficients can be written on dotted edges connecting the smoothed curves as depicted.



The value of a totally smoothed diagram is then the product of the elements on the dashed edges with δ^k where k is the number of solid components. These values are then summed over the set of all totally smoothed diagrams and multiplied by the writhe correction factor w^{n-p} to yield the contribution $\beta(f)$ of the coloring $f \in \mathrm{Hom}(\mathcal{B}(L), X)$ to the invariant $\Phi_X^{\beta}(L)$.

Proposition 1. *Let X be a finite biquandle and let β and β' be X-brackets over R defined by maps $A, B : X \times X \to R^\times$ and $A', B' : X \times X \to R^\times$ respectively. If there is an invertible scalar $\alpha \in R^\times$ such that for all $x, y \in X$ we have*

$$A_{x,y} = \alpha A'_{x,y} \quad \text{and} \quad B_{x,y} = \alpha B'_{x,y}$$

then the link invariants defined by β and β' are equal.

Proof. First, we note that

$$\delta' = -A'_{x,y}B'^{-1}_{x,y} - A'^{-1}_{x,y}B'_{x,y} = -(\alpha A_{x,y})(\alpha B_{x,y})^{-1} - (\alpha A^{-1}_{x,y})\alpha B_{x,y} = -A_{x,y}B^{-1}_{x,y} - A^{-1}_{x,y}B_{x,y} = \delta$$

and

$$w' = A'_{x,x}\delta + B'_{x,x} = \alpha A_{x,x}\delta + \alpha B_{x,x} = \alpha w.$$

Then for any link diagram L with j positive crossings and k negative crossings, the state sum with β' equals that with β multiplied by α^{j-k} at every crossing. Then the contribution $\beta'(f)$ equals $\beta(f)$ multiplied by α^{k-k}, then multiplied by α^{k-j} in the writhe-correction factor $(w')^{k-j}$; hence, the powers of α cancel and we have $\beta(f) = \beta'(f)$, whence $\Phi^{\beta}_{x}(L) = \Phi^{\beta'}_{x}(L)$ for all classical and virtual knots and links L. □

Example 9. The simplest non-trivial biquandle is the constant action biquandle on $X = \{1, 2\}$ with operation matrix

$$\left[\begin{array}{cc|cc} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right].$$

The counting invariant $\Phi^{\mathbb{Z}}_X(L)$ with respect to this biquandle is 0 if L is a virtual link with any component containing an odd number of crossing points and is 2^c where c is the number of components of L otherwise. Our `python` computations reveal biquandle bracket structures on X with coefficients in $\mathbb{Z}_5$ including

$$\left[\begin{array}{cc|cc} 1 & 3 & 4 & 2 \\ 4 & 1 & 1 & 4 \end{array}\right].$$

The Hopf link has four X-labelings and fundamental biquandle bracket value

$$\phi = A_{x,y}A_{y,x}\delta^2 + B_{x,y}A_{y,x}\delta + A_{x,y}B_{y,x}\delta + B_{x,y}B_{y,x}\delta^2.$$

Then we have $\delta = 2$, $w = 1$ and

x	y	ϕ
1	1	$1(1)(2^2) + 1(4)(2) + 4(1)(2) + 4(4)(2^2) = 4 + 3 + 3 + 4 = 4$
1	2	$3(4)(2^2) + 2(4)(2) + 3(1)(2) + 2(1)(2^2) = 3 + 1 + 1 + 3 = 3$
2	1	$4(3)(2^2) + 1(3)(2) + 4(2)(2) + 1(2)(2^2) = 3 + 1 + 1 + 3 = 3$
2	2	$1(1)(2^2) + 1(4)(2) + 4(1)(2) + 4(4)(2^2) = 4 + 3 + 3 + 4 = 4$

Then the Hopf link has biquandle bracket invariant

$$\Phi^{B}_X(L) = 2u^3 + 2u^4$$

while the unlink of two components U_2 has invariant value

$$\Phi^{B}_X(U_2) = 4u^4.$$

Example 10. Let X be any finite biquandle and R be any commutative ring with identity. For any invertible element $t \in R$, the maps $A(x, y) = t$, $B(x, y) = t^{-1}$ define a biquandle bracket β_t called a *constant biquandle bracket*. For any link L, the biquandle bracket invariant with respect to β_t is $\Phi^{\beta}_X(L) = \Phi^{\mathbb{Z}}_X(L)u^{K_L(t)}$ where $K_L(t)$ is the Kauffman bracket polynomial of L evaluated at t.

Example 11. More generally, an X-bracket in which for all $x, y \in X$ we have $A(x, y) = A$ and $B(x, y) = B$ where $A+A^{-1} = B+B^{-1}$ defines a biquandle bracket $\beta_{A,B}$ satisfying $\Phi_X^{\beta_{A,B}}(L) = \Phi_X^{\mathbb{Z}}(L)u^{K_L(-A^2B^{-1},A+A^{-1})}$ where $K(a, z)$ is the Kauffman 2-variable polynomial. Similarly, a biquandle bracket $\beta_{A,B}$ with $A(x, y) = A$ and $B(x, y) = B$ with $-A^{-1}B - AB^{-1} = 1$ yields $\Phi_X^{\beta_{a,z}}(L) = \Phi_X^{\mathbb{Z}}(L)u^{H_L(A^{-1},A^{-1}B-AB^{-1})}$ where $H(a, z)$ is the HOMFLY-PT polynomial [11].

Example 12. Let X be a finite biquandle, G an abelian group, and $\psi \in H^2(X; G)$ an element of the second cohomology of X with G coefficients, i.e., a function $\psi : X \times X \to G$ satisfying for all $x, y, z \in X$

$$\psi(x, y)\psi(y, z)\psi(x \,\overline{\triangleright}\, y, z \,\underline{\triangleright}\, y) = \psi(x, z)\psi(y \,\overline{\triangleright}\, x, z \,\overline{\triangleright}\, x)\psi(x \,\underline{\triangleright}\, z, y \,\underline{\triangleright}\, z)$$

and $\psi(x, x) = 1$ (see [4] for instance). Then setting $A(x, y) = B(x, y) = \psi(x, y)$ defines a biquandle bracket with $R = \mathbb{Z}[G]$. Indeed, every biquandle bracket with $A(x, y) = B(x, y)$ for all $x, y \in X$ arises in this way, since the biquandle bracket conditions with $A_{x,y} = B_{x_y}$ reduce to $\delta = -2$, $w = -1$ and the 2-cocycle condition

$$A_{x,y}A_{y,z}A_{x \,\underline{\triangleright}\, y, z \,\overline{\triangleright}\, y} = A_{x,z}A_{y \,\overline{\triangleright}\, x, z \,\overline{\triangleright}\, x}A_{x \,\underline{\triangleright}\, z, y \,\underline{\triangleright}\, z}.$$

The biquandle bracket invariant in this case satisfies

$$\Phi_X^{\beta}(L) = \Phi_X^{\psi}(L)K_L(1)$$

where $K_L(1)$ is the Kauffman bracket polynomial of L evaluated at $A = 1$.

Proposition 2. *Let X be a finite biquandle, R be a commutative ring and $C : X \to R^{\times}$ be a map as in example 7, and let $C : X \times X \to \mathbb{R}^{\times}$ be the biquandle bracket defined by setting both A and B equal to*

$$C(x, y) = C(x)C(y)^{-1}C(x \,\underline{\triangleright}\, y)^{-1}C(y \,\overline{\triangleright}\, x).$$

Then for any biquandle bracket β defined by $A, B : X \times X \to R^{\times}$, the maps

$$A'(x, y) = A(x, y)C(x, y) \quad \text{and} \quad B'(x, y) = B(x, y)C(x, y)$$

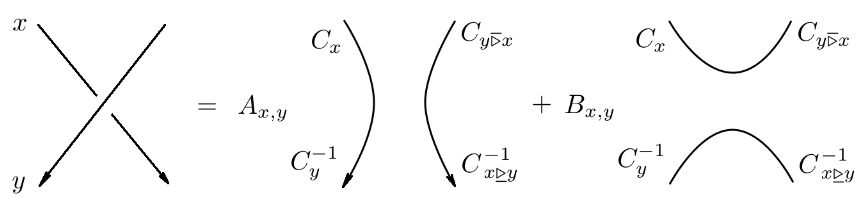
define a biquandle bracket $C\beta$ with $\delta = -A_{x,y}B_{x,y}^{-1} - A_{x,y}^{-1}B_{x,y}$ and we have $\Phi_X^{\beta} = \Phi_X^{C\beta}$.

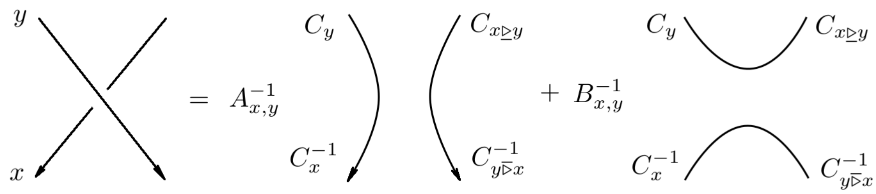
Proof. In $C\beta$, the invertible quantity

$$C(x)C(y \,\overline{\triangleright}\, x)C(z)^{-1}C(y \,\underline{\triangleright}\, z)^{-1}C((x \,\underline{\triangleright}\, y) \,\underline{\triangleright}\, (z \,\overline{\triangleright}\, y))^{-1}C((z \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (x \,\underline{\triangleright}\, y))$$

factors out of each term on both sides of the equations in biquandle bracket axiom (iii), so $C\beta$ is a biquandle bracket provided β is.

To see that β and $C\beta$ define the same invariant, note that we can picture $C\beta$ as including factors of $C(x)$, $C(y \,\overline{\triangleright}\, x)$, $C(y)^{-1}$, $C(x \,\underline{\triangleright}\, y)^{-1}$ and on the initial and terminal ends of the semiarc respectively along with the $A_{x,y}$ and $B_{x,y}$ coefficients as shown.





Then we observe that over any complete link diagram, the C factors match up in canceling pairs along each semiarc, so the value of each state of an X-colored link in $\Phi_X^{C\beta}$ is the same as in Φ_X^{β}. $\square$

For biquandle brackets β representing biquandle 2-cocycles, C is a coboundary and β and $C\beta$ are cohomologous; however, Proposition 2 holds even for biquandle brackets β not representing cocycles. Thus, it is tempting to define β and $C\beta$ to be "cohomologous" regardless of whether β is a cocycle; however, we will settle for the following:

Definition 4. Two X-brackets β and β' over R are *C-equivalent* if there is a map $C : X \to R^{\times}$ such that for all $x, y \in X$, we have

$$\begin{array}{rcl} A'(x,y) & = & A(x,y)C(x)C(y)^{-1}C(x\,\underline{\triangleright}\,y)^{-1}C(y\,\overline{\triangleright}\,x) \text{ and} \\ B'(x,y) & = & B(x,y)C(x)C(y)^{-1}C(x\,\underline{\triangleright}\,y)^{-1}C(y\,\overline{\triangleright}\,x). \end{array}$$

Corollary 3. *C-equivalent X-brackets define the same invariant Φ_X^{β}.*

In [12], *quantum enhancements* of the counting invariant with respect to involutory biquandles X were defined as functors from the category of X-labeled unoriented tangles to an R-module category. Biquandle brackets provide examples of quantum enhancements as defined in [12] in the following way: Given a biquandle bracket β, define

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & A_{11} & -B_{11} & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 \\ -B_{11}^{-1} \\ A_{11}^{-1} \\ 0 \end{bmatrix}.$$

Then the biquandle bracket skein relation yields X-labeled R-matrices $X_{x,y}^{\pm 1}$:

$$\begin{array}{rcl} X_{x,y} & = & A_{x,y}(I \otimes I) + B_{x,y}(UN) \\ & = & \begin{bmatrix} A_{x,y} & 0 & 0 & 0 \\ 0 & 0 & B_{x,y} & 0 \\ 0 & B_{x,y} & A_{x,y} - A_{x,y}^{-1}B_{x,y}^2 & 0 \\ 0 & 0 & 0 & A_{x,y} \end{bmatrix}. \end{array}$$

See [14] for more.

Example 13. The biquandle bracket in example 9 corresponds to quantum weight over $\mathbb{Z}_5$ given by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

$$X_{1,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad X_{1,2} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

$$X_{2,1} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad \text{and} \quad X_{2,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In particular, this quantum enhancement is an example of a *strongly heterogeneous* quantum weight as defined in the questions in [12], since $X_{1,2}$ is not a classical R-matrix.

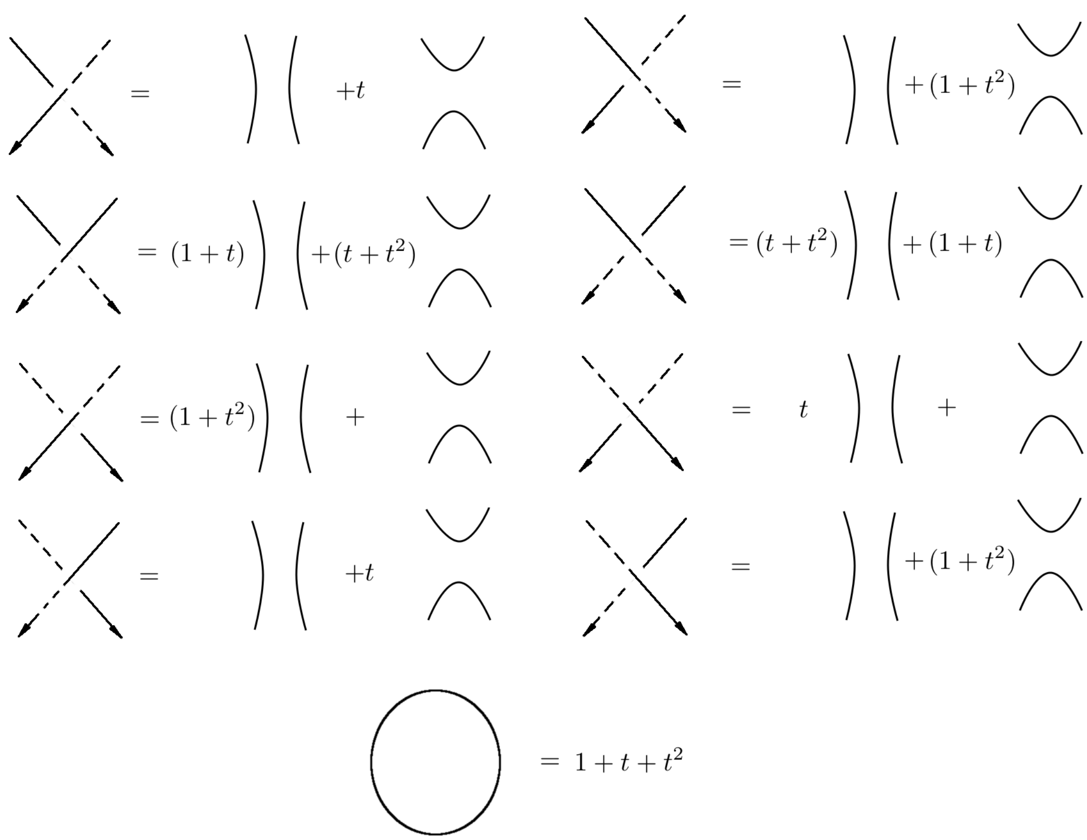
Example 14. Let X be the biquandle defined by the operation matrix

$$\left[\begin{array}{cc|cc} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

and let $R = \mathbb{F}_8 = \mathbb{Z}_2[t]/(1+t+t^3)$ be the Galois field of eight elements. That is, R is the ring of polynomials in one variable with $\mathbb{Z}_2$ coefficients with the rule that $t^3 = 1+t$. Then our `python` computations reveal that

$$\left[\begin{array}{cc|cc} 1 & 1+t & t & t+t^2 \\ 1+t^2 & 1 & 1 & t \end{array}\right]$$

defines a biquandle bracket. We can describe this one without explicitly referencing biquandles in the following way: Given any oriented link L of c components, find the the 2^c ways to color the semiarcs of L alternately solid and dotted going around each component. Then for each such coloring, expand using the following skein relations.



Finally, multiply by the writhe normalization factor $t^p(1+t^2)^n$ where p and n are the numbers of positive and negative crossings respectively.

We computed this invariant for all prime classical knots with up to eight crossings, all prime classical links with up to seven crossings, and all prime virtual knots with up to four classical crossings as found in the tables at the knot atlas [1]. The results are collected in the tables below. We list the multiset version of the invariant for ease of reading. We start with the prime classical knots:

$\Phi_X^\beta(K)$	K
$\{2\times 1\}$	$5_2, 7_5, 8_{10}, 8_{11}, 8_{13}, 8_{17}$
$\{2\times t\}$	$3_1, 6_2, 8_9$
$\{2\times 1+t\}$	$4_1, 7_1, 7_4, 8_5, 8_{14}$
$\{2\times t^2\}$	$6_1, 6_3, 7_2, 7_3, 8_7, 8_{21}$
$\{2\times 1+t^2\}$	$7_7, 8_2, 8_3, 8_4, 8_8, 8_{19}, 8_{20}$
$\{2\times t+t^2\}$	$8_1, 8_6, 8_{12}, 8_{16}, 8_{18}$
$\{2\times 1+t+t^2\}$	Unknot, $5_1, 7_6, 8_{15}$

For prime classical links with up to seven crossings, we have

$\Phi_X^\beta(L)$	L
$\{2\times 1, 2\times t\}$	$L6a2$
$\{2\times t, 2\times 1+t^2\}$	$L7a6$
$\{2\times 1, 2\times t+t^2\}$	$L6a1$
$\{2\times 1, 2\times 1+t+t^2\}$	$L7a5$
$\{2\times t^2, 2\times 1+t+t^2\}$	$L7a2, L7n1$
$\{2\times 1+t^2, 2\times t+t^2\}$	$L2a1$
$\{2\times 1+t^2, 2\times 1+t\}$	$L4a1$
$\{2\times t+t^2, 2\times 1+t+t^2\}$	$L6a3$
$\{4\times 0\}$	$L7n2$
$\{4\times t^2\}$	$L5a1$
$\{4\times t+t^2\}$	$L7a1, L7a3, L7a4$
$\{2\times 1, 6\times t+t^2\}$	$L6a4, L6n1$
$\{2\times t^2, 6\times 1+t^2\}$	$L6a5$
$\{2\times t, 6\times t+t^2\}$	$L7a7$

and for prime virtual knots with up to four classical crossings we have

$\Phi_X^\beta(K)$	K
$\{2\times 0\}$	2.1, 3.5, 4.2, 4.6, 4.8, 4.12, 4.17, 4.28, 4.32, 4.51, 4.58, 4.71, 4.75, 4.89, 4.105
$\{2\times 1\}$	4.23, 4.41, 4.65, 4.79
$\{2\times t\}$	3.6, 4.15, 4.16, 4.20, 4.22, 4.34, 4.40, 4.52, 4.60, 4.64, 4.82, 4.87, 4.92, 4.94
$\{2\times 1+t\}$	4.9, 4.10, 4.29, 4.31, 4.37, 4.48, 4.50, 4.57, 4.61, 4.69, 4.70, 4.78, 4.86, 4.90, 4.99, 4.108
$\{2\times t^2\}$	3.3, 4.4, 4.5, 4.11, 4.18, 4.25, 4.30, 4.33, 4.38, 4.39, 4.43, , 4, 44, 4.45, 4.49, 4.54, 4.62, 4.63, 4.74, 4.80, 4.83, 4.84, 4.88, 4.91, 4.95, 4.100, 4.101, 4.104
$\{2\times 1+t^2\}$	4.1, 4.3, 4.7, 4.21, 4.24, 4.36, 4.53, 4.68, 4.73
$\{2\times t+t^2\}$	3.2, 3.4, 4.27, 4.81
$\{2\times 1+t+t^2\}$	3.1, 3.7, 4.13, 4.19, 4.26, 4.35, 4.42, 4.46, 4.47, 4.55, 4.56, 4.59, 4.66, 4.67, 4.72, 4.76, 4.77, 4.85, 4.93, 4.96, 4.97, 4.98, 4.102, 4.103, 4.106, 4.107

We note that:

- Φ_X^β distinguishes the right- and left-hand trefoils, with invariant values of $\{2\times t\}$ and $\{2\times 0\}$ respectively and hence can distinguish mirror images,
- Φ_X^β distinguishes the Square knot from the Granny knot with invariant values of $\{2\times t+t^2\}$ and $\{2\times 0\}$ respectively, so Φ_X^β is not determined by the knot group, and

- $\Phi_X^{\beta}(10_{132}) = \{2\times t+t^2\} \neq \{2\times 1+t+t^2\} = \Phi_X^{\beta}(5_1)$ and hence Φ_X^{β} is not determined by the HOMFLY-PT, Jones, or Alexander polynomials.

4 Quandle Brackets

Let X be a *quandle*, that is, a biquandle with $x \,\overline{\triangleright}\, y = x$ for all $x, y \in X$. An X-bracket in this case is called a *quandle bracket.*

Proposition 4. *If X is a quandle and R is a commutative ring, then maps $A, B : X\times X \to R^{\times}$ defining a quandle bracket must satisfy the mixed cocycle conditions*

$$\begin{array}{rcll} A_{x,y}A_{x\,\underline{\triangleright}\,y,z} & = & A_{x,z}A_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} & (i)\\ A_{x,y}B_{x\,\underline{\triangleright}\,y,z} & = & B_{x,z}A_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} & (ii)\\ B_{x,y}A_{x\,\underline{\triangleright}\,y,z} & = & A_{x,z}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} & (iii)\\ B_{x,y}B_{x\,\underline{\triangleright}\,y,z} & = & B_{x,z}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} & (iv). \end{array}$$

Proof. Suppose our biquandle X is a quandle, i.e., $x \,\overline{\triangleright}\, y = x$ for all $x, y \in X$. Then the first three biquandle bracket conditions from the Reidemeister III move reduce to

$$\begin{array}{rcl} A_{x,y}A_{y,z}A_{x\,\underline{\triangleright}\,y,z} & = & A_{x,z}A_{y,z}A_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z}\\ A_{x,y}B_{y,z}B_{x\,\underline{\triangleright}\,y,z} & = & B_{x,z}B_{y,z}A_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z}\\ B_{x,y}A_{y,z}B_{x\,\underline{\triangleright}\,y,z} & = & B_{x,z}A_{y,z}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} \end{array} \quad\Rightarrow\quad \begin{array}{rcl} A_{x,y}A_{x\,\underline{\triangleright}\,y,z} & = & A_{x,z}A_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z}\\ A_{x,y}B_{x\,\underline{\triangleright}\,y,z} & = & B_{x,z}A_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z}\\ B_{x,y}B_{x\,\underline{\triangleright}\,y,z} & = & B_{x,z}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} \end{array}$$

yielding (i),(ii) and (iv). Then the remaining biquandle bracket equations say

$$\begin{array}{rcl} A_{y,z}(A_{x,y}B_{x\,\underline{\triangleright}\,y,z} - A_{x,z}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z}) & = & B_{y,z}(A_{x,z}A_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} + \delta A_{x,z}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} + B_{x,z}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z})\\ A_{y,z}(B_{x,y}A_{x\,\underline{\triangleright}\,y,z} - B_{x,z}A_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z}) & = & -B_{y,z}(A_{x,y}A_{x\,\underline{\triangleright}\,y,z} + \delta B_{x,y}A_{x\,\underline{\triangleright}\,y,z} + B_{x,y}B_{x\,\underline{\triangleright}\,y,z}) \end{array}$$

which then implies

$$A_{y,z}(B_{x,y}A_{x\,\underline{\triangleright}\,y,z} - A_{x,z}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z}) = \delta B_{y,z}(A_{x,z}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} - A_{x,y}B_{x\,\underline{\triangleright}\,y,z})$$

so we have

$$(B_{x,y}A_{x\,\underline{\triangleright}\,y,z} - A_{x,z}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z})(A_{x,y} + \delta B_{y,z}) = 0.$$

Then

$$A_{y,z} + \delta B_{y,z} = A_{y,z} + (-A_{yz}B_{y,z}^{-1} - A_{y,z}^{-1}B_{y,z})B_{y,z} = -A_{y,z}^{-1}B_{y,z}^2$$

is a unit in R, so $B_{x,y}A_{x\,\underline{\triangleright}\,y,z} - A_{x,z}B_{x\,\underline{\triangleright}\,z,y\,\underline{\triangleright}\,z} = 0$ as required. □

We note that the converse to proposition 4 is not true – the mixed cocycle conditions are necessary but not sufficient conditions for maps $A : X \times X \to R$ to define a quandle bracket, as the next example demonstrates.

Example 15. Consider the trivial quandle on two elements, $T_2 = \{1, 2\}$ with $x \,\underline{\triangleright}\, y = x \,\overline{\triangleright}\, y = x$. The maps $A, B : X \times X \to \mathbb{Z}_3$ defined by

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1\\ 1 & 2 & 1 & 2 \end{array}\right]$$

satisfy all four mixed cocycle conditions and also the conditions that

$$\delta = -A_{x,y}B_{x,y}^{-1} - A_{x,y}^{-1}B_{x,y} = -2 = 1$$

and

$$w = 2 = A_{x,x}\delta + B_{x,x}$$

for all $x, y \in X$; however, this is not a biquandle bracket since $A_{1,2}A_{2,2}B_{12} = 2$ but

$$A_{1,2}B_{2,2}A_{1,2} + A_{1,2}A_{2,2}B_{1,2} - 2A_{1,2}B_{2,2}B_{12} + B_{1,2}B_{2,2}B_{1,2} = 4 = 1 \neq 2$$

so the fourth equation in biquandle bracket axiom (iii) is not satisfied.

5 Questions

We end with some questions for future research. This is second paper in an ongoing series on quantum enhancements; future papers are underway extending the present results to knotted surface and virtual knots in various ways.

What exactly is the relationship between biquandle and brackets biquandle cohomology? Is there a generalized theory of biquandle cohomology which includes those biquandle brackets which are not biquandle cocycles in the traditional sense? Are there quantum enhancements which do not arise from biquandle brackets? What Khovanov homology-style categorifications of biquandle bracket invariants are possible? What about biquandle-colored skein modules?

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